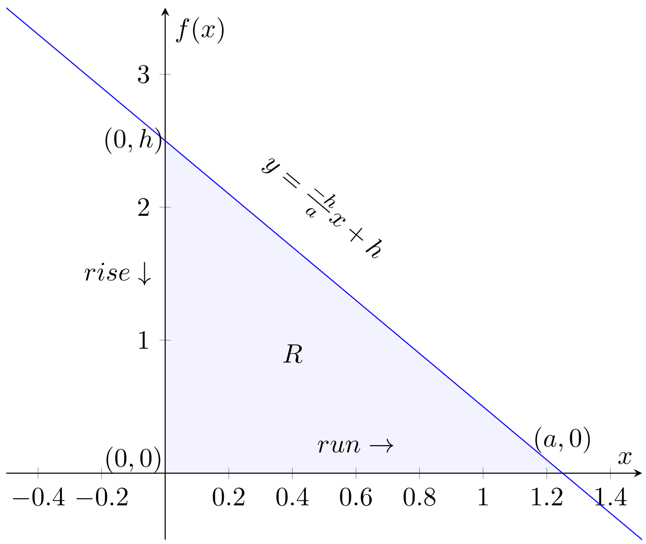
1 WS 2 (Sept 5) solutions

• 1. Let $a > 0$ and $h > 0$, consider the triangular region R whose vertices are $(0,0), (a,0)$ and $(0,h)$.

a) Find a formula for the line that represents the hypotenuse of the triangle.

First we draw our region:



We'll use slope intercept form since we have the y-intercept and we can find the slope.

Note that the slope of this line is $m = \frac{rise}{run} = \frac{-h}{a}$ with y-intercept $b = h$, so our equation for the line should be $y = mx + b = \frac{-h}{a}x + h$.

b) Show that the center of gravity of R is $(\frac{a}{3}, \frac{h}{3})$.

To find the center of gravity $(\bar{x}, \bar{y})$ of R, the region bounded by f and g on $[x_0, x_1]$, we'll use the formulae $\bar{x} = \frac{M_y}{A}$, $\bar{y} = \frac{M_x}{A}$, where A is the area of R, and

$$M_y = \int_{x_0}^{x_1} x(f(x) - g(x))dx,$$
$$M_x = \frac{1}{2}\int_{x_0}^{x_1} (f(x))^2 - (g(x))^2 dx.$$

For us, $f(x) = \frac{-h}{a}x + h$, $g(x) = 0$ on $[x_0, x_1] = [0, a]$, and the area A of R is $\frac{1}{2}(base)(height) = \frac{1}{2}ah$ (we could also integrate f over $[0, a]$ to get the same). Thus

$$M_y = \int_0^a x(\frac{-h}{a}x + h)dx$$
$$= \int_0^a (\frac{-h}{a}x^2 + hx)dx$$
$$= [\frac{-h}{a}\frac{x^3}{3} + h\frac{x^2}{2}]_0^a = \frac{-h}{a}\frac{a^3}{3} + h\frac{a^2}{2} = \frac{-ha^2}{3} + \frac{ha^2}{2}$$
$$= \frac{ha^2}{6}$$

,

meanwhile

$$M_x = \frac{1}{2}\int_0^a (\frac{-h}{a}x + h)^2 dx$$
$$= \frac{1}{2}\int_0^a (\frac{h^2}{a^2}x^2 - 2\frac{h^2}{a}x + h^2)dx$$
$$= \frac{1}{2}[\frac{h^2}{a^2}\frac{x^3}{3} - 2\frac{h^2}{a}\frac{x^2}{2} + h^2 x]_0^a$$

$$= \frac{1}{2}(\frac{h^2}{a^2}\frac{a^3}{3} - 2\frac{h^2}{a}\frac{a^2}{2} + h^2 a)$$
$$= \frac{1}{2}(\frac{h^2 a}{3} - h^2 a + h^2 a) = \frac{h^2 a}{6}$$

Then,

$$\bar{x} = \frac{M_y}{A} = \frac{\frac{ha^2}{6}}{\frac{ha}{2}}$$
$$= \frac{ha^2}{6} \cdot \frac{2}{ha} = \frac{a}{3},$$

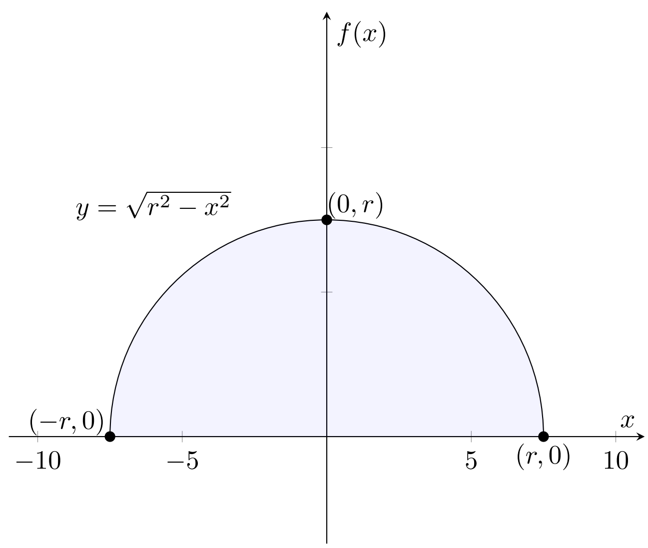
meanwhile

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{h^2 a}{6}}{\frac{ha}{2}}$$
$$= \frac{h^2 a}{6} \cdot \frac{2}{ha} = \frac{h}{3}.$$

• 2. Let $r > 0$ and let R be the semicircular region bounded below by the x-axis and above by the circle $x^2 + y^2 = r^2$, that is, $x^2 + y^2 = r^2$ with $y \geq 0$.

a) Find the center of gravity $(\bar{x}, \bar{y})$ of R.

First we draw our region:



Again, since we need to find $\bar{x} = \frac{M_y}{A}$, $\bar{y} = \frac{M_x}{A}$, we use our formulae for moments, together with the area of a semicircle $A = \frac{1}{2}\pi r^2$ since we can't integrate $\sqrt{r^2 - x^2}$ to find A (yet).

$$M_y = \int_{x_0}^{x_1} x(f(x) - g(x))dx,$$
$$M_x = \frac{1}{2}\int_{x_0}^{x_1} (f(x))^2 - (g(x))^2 dx.$$

For us, $f(x) = \sqrt{r^2 - x^2}$, $g(x) = 0$ on $[x_0, x_1] = [-r, r]$ so:

$$M_y = \int_{x_0}^{x_1} x(f(x) - g(x))dx = \int_{-r}^{r} x(\sqrt{r^2 - x^2})dx.$$

Let $u = r^2 - x^2$ so that $du = (-2x)dx \to \frac{-1}{2}du = xdx$ (we're treating r as constant here), and we change the bound $x = -r$ to $u = r^2 - (-r)^2 = 0$ and $x = r$ to $u = r^2 - r^2 = 0$ so from $u = 0$ to $u = 0$ in our bounds. Then, we have

$$\int_{x=-r}^{x=r} x(\sqrt{r^2 - x^2})dx = \frac{-1}{2}\int_{u=0}^{u=0} \sqrt{u}du = 0.$$

The last equality follows from the property of integrals $\int_a^a f(x)dx = F(a) - F(a) = 0$ where $F'(x) = f(x)$ (F is the antiderivative of f).

This tells us that $\bar{x} = \frac{M_y}{A} = \frac{0}{A} = 0$.

To find $\bar{y}$, let's find M_x:

$$M_x = \frac{1}{2}\int_{x_0}^{x_1}(f(x))^2 - (g(x))^2 dx$$
$$= \frac{1}{2}\int_{-r}^{r}(\sqrt{r^2 - x^2})^2)dx$$
$$\frac{1}{2}\int_{-r}^{r}(r^2 - x^2)dx$$
$$= \frac{1}{2}[r^2 x - \frac{x^3}{3}]_{-r}^{r} = \frac{1}{2}[(r^3 - \frac{r^3}{3}) - ((-r)^3 - \frac{(-r)^3}{3})] = \frac{1}{2}(2 \cdot \frac{2}{3})r^3 = \frac{2r^3}{3}.$$

Then, $\bar{y} = \frac{M_x}{A} = \frac{\frac{2r^3}{3}}{\frac{\pi r^2}{2}} = \frac{2r^3}{3} \cdot \frac{2}{\pi r^2} = \frac{4r}{3\pi}$.

Thus our center of gravity is $(\bar{x}, \bar{y}) = (0, \frac{4r}{3\pi})$.

b) Find the radius r for which $(\bar{x}, \bar{y}) = (0, \pi)$.

We simply set $(\bar{x}, \bar{y}) = (0, \frac{4r}{3\pi}) = (0, \pi)$, so $\frac{4r}{3\pi} = \pi$ and solve for r:

$$4r = 3\pi^2, \text{ so } r = \frac{3\pi^2}{4}.$$

2 Quiz 2 (Sept 5) solutions

• 1. A bucket of cement weighing 200kg is hoisted by means of a windlass from the ground to the tenth story of an office building, 80m above the ground.

a) If the weight of the rope is negligible, find the work W required to make the lift.

Since the downward force of gravity on the bucket of cement is constant and the weight of the rope is negligible,

$$F_{system}(y) = F_{bucket}(y) = 200kg \cdot 9.8\frac{m}{s^2} = 1,960N$$
[using that $F = ma =$ (mass)(acceleration)].

We then simply have:

$$W = F_{bucket} \cdot d = 1,960N \cdot 80m = 156,800 \text{ J (joules) (not from Euphoria) (relax)}$$

b) Assume that a chain weighing $1kg$ per meter is used in (a) instead of the lightweight rope. Find the work W required to make the lift. (Hint: As the bucket is raised, the length of chain that must be lifted decreases).

Now, we need to add the force of gravity on the chain which is changing as y meters of chain hangs off the building, so that

$$y \text{ meters} \cdot 1\frac{kg}{meter} = y \text{ kilograms} = m_{chain}$$

is the mass of the chain being acted on by gravity.

Then $F_{chain} = m_{chain} \cdot g = y$ kilograms $\cdot\, 9.8\frac{m}{s^2} = 9.8yN$, so $F_{system}(y) = F_{bucket}(y) + F_{chain}(y) = 1920N + 9.8yN$. Then,

$$\begin{aligned}W &= \int_{y=0m}^{y=80m} F_{system}(y)dy \\ &= \int_{y=0m}^{y=80m} (1920 + 9.8y)dy \\ &= [1920y + 9.8\tfrac{y^2}{2}]_{0m}^{80m} \\ &= 1920N \cdot 80m + 9.8\tfrac{m}{s^2} \cdot \tfrac{80^2 m^2}{2} \\ &= 156,800 + 9.8 \cdot 3200 \text{ J} = 156,800 + 31,360 \text{ J} \\ &= 188,160 \text{ J.}\end{aligned}$$

OR

• 2. When a certain spring is expanded $10cm$ from its natural position and held fixed, the force necessary to hold it is $4 \cdot 10^6$ dynes. Find the work required to stretch the spring an additional $10cm$ (1 dyne $= 1cm -$ gram-second $= 10^{-5}N$).

Well, the force $F(x)$ needed in order to hold a spring extended x units beyond its natural length is $F(x) = kx$ for some spring constant k. We need to find this k:

$$F(10cm) = 4 \cdot 10^6 \text{dynes} = k \cdot 10cm$$
$$\implies k = 4 \cdot 10^5 (\text{gram-seconds}).$$

Now we integrate $F(x) = 4 \cdot 10^5 x$ from $x = 10$ to $x = 20$ since the work required in order to stretch a spring from a units extended to b units is given by the formula:

$$\begin{aligned}W &= \int_a^b F(x)dx \\ \implies W &= \int_{10cm}^{20cm} (4 \cdot 10^5 \text{gram- seconds})x dx \\ &= 4 \cdot 10^5 (\text{gram- seconds})[\tfrac{x^2}{2}]_{10cm}^{20cm} \\ &= 4 \cdot 10^5 (\text{gram- seconds})(\tfrac{20^2 cm^2}{2} - \tfrac{10^2 cm^2}{2}) \\ &= 4 \cdot 10^5 (\text{gram-seconds})\tfrac{400cm^2 - 100cm^2}{2} \\ &= (4 \cdot 10^5 \text{gram-seconds}) \cdot 150cm^2 \\ &= 6 \cdot 10^7 (cm - \text{gram-seconds}) \cdot cm \\ &= 6 \cdot 10^7 \text{dyne} - cm = 6 \cdot 10^7 \text{ergs} \\ &(1 \text{ dyne} - cm = 1 \text{ erg}).\end{aligned}$$